

CORNERS OF CUNTZ-KRIEGER ALGEBRAS

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ABSTRACT. We show that if A is a unital C^* -algebra and B is a Cuntz-Krieger algebra for which $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$, then A is a Cuntz-Krieger algebra. Consequently, corners of Cuntz-Krieger algebras are Cuntz-Krieger algebras.

1. INTRODUCTION

The Cuntz-Krieger algebras were introduced by J. Cuntz and W. Krieger in 1980, [7], as C^* -algebras arising from dynamical systems. This class of C^* -algebras has since shown up in several contexts, including the classification program as the Cuntz-Krieger algebras with finitely many ideals are examples of non-simple purely infinite C^* -algebras. It has been known since M. Enomoto and Y. Watatani introduced graph algebras in 1980 in [10], that Cuntz-Krieger algebras are the graph algebras arising from finite graphs with no sinks and no sources (see also [12]), but no characterization in terms of outer properties has been established for the Cuntz-Krieger algebras.

We show in Theorem 3.7 that the Cuntz-Krieger algebras are the graph algebras arising from finite graphs with no sinks, and conclude that a graph algebra is a Cuntz-Krieger algebra if and only if it is unital and the rank of its K_0 -group equals the rank of its K_1 -group.

Using this, we show in Theorem 4.8, that if a unital C^* -algebra is stably isomorphic to a Cuntz-Krieger algebra, then it is isomorphic to a Cuntz-Krieger algebra.

As a corollary to Theorem 4.8, we see that corners of Cuntz-Krieger algebras are Cuntz-Krieger algebras, cf. Corollary 4.10. It is quite surprising that the class of Cuntz-Krieger algebras has this permanence property since the larger class of graph algebras do not (as the graph algebra $M_{2^\infty} \otimes \mathbb{K}$ provides a counter-example). Moreover, this shows that corners of Cuntz-Krieger algebras are semiprojective, Corollary 4.11, as Cuntz-Krieger algebras are semiprojective. Our results also show that unital corners of a stabilized Cuntz-Krieger algebra are semiprojective since a stabilized Cuntz-Krieger algebra is semiprojective. It was conjectured by B. Blackadar in [4, Conjecture 4.4] that full corners of a semiprojective C^* -algebra are semiprojective.

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He showed in [4, Proposition 2.7] that full unital corners of semiprojective C^* -algebras are semiprojective. Recently, S. Eilers and T. Katsura in [9] showed that every corner of a unital graph C^* -algebra which is semiprojective is also semiprojective. Corollary 4.11 is a special case of their results since every Cuntz-Krieger algebra is isomorphic to a unital semiprojective graph C^* -algebra. Semiprojectivity is easy in our case since the graphs are finite, thus we do not need any results from [9].

2. DEFINITIONS

Definition 2.1. Let $E = (E^0, E^1, s_E, r_E)$ be a (countable directed) graph. A Cuntz-Krieger E -family is a set of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a set $\{s_e \mid e \in E^1\}$ of partial isometries satisfying the following conditions:

- (CK0) $s_e^* s_f = 0$ if $e, f \in E^1$ and $e \neq f$,
- (CK1) $s_e^* s_e = p_{r_E(e)}$ for all $e \in E^1$,
- (CK2) $s_e s_e^* \leq p_{s_E(e)}$ for all $e \in E^1$, and,
- (CK3) $p_v = \sum_{e \in s_E^{-1}(v)} s_e s_e^*$ for all $v \in E^0$ with $0 < |s_E^{-1}(v)| < \infty$.

The *graph algebra* $C^*(E)$ is defined as the universal C^* -algebra given by these generators and relations.

Definition 2.2. A graph C^* -algebra given by a finite graph with no sinks and no sources is called a *Cuntz-Krieger algebra*.

Although this is not the original definition of a Cuntz-Krieger algebra, due to J. Cuntz and W. Krieger in [7], it has been shown to be equivalent to it. I.e., every Cuntz-Krieger algebra is isomorphic to $C^*(E)$ for some finite graph E with no sinks and no sources, and conversely for any finite graph E with no sinks and no sources, $C^*(E)$ is isomorphic to a Cuntz-Krieger algebra.

3. GRAPH C^* -ALGEBRAS OVER FINITE GRAPHS WITH NO SINKS

Let E be a graph, let $v_0 \in E^0$ that is not a source and let $e_0 \in r_E^{-1}(v_0)$. Define a graph F_1 as follows:

$$\begin{aligned} F_1^0 &= E^0 \cup \{v_1, v_2, \dots, v_n\} \\ F_1^1 &= E^1 \cup \{e_1, e_2, \dots, e_n\} \end{aligned}$$

where r_{F_1} and s_{F_1} extends r_E and s_E respectively and $r_{F_1}(e_i) = v_{i-1}$ and $s_{F_1}(e_i) = v_i$.

Define a graph F_2 as follows:

$$\begin{aligned} F_2^0 &= E^0 \cup \{v_1, v_2, \dots, v_n\} \\ F_2^1 &= (E^1 \setminus \{e_0\}) \cup \{e_1, e_2, \dots, e_{n+1}\} \end{aligned}$$

where r_{F_2} and s_{F_2} extends r_E and s_E respectively, $r_{F_2}(e_i) = v_{i-1}$ and $s_{F_2}(e_i) = v_i$ for $i = 1, \dots, n$, and $r_{F_2}(e_{n+1}) = v_n$ and $s_{F_2}(e_{n+1}) = s_E(e_0)$.

Proposition 3.1. *Let E , F_1 , and F_2 be as above. Then $C^*(F_1) \cong C^*(F_2)$.*

Proof. Let $\{s_e, p_v : e \in F_2^1, v \in F_2^0\}$ be a universal Cuntz-Krieger F_2 -family generating $C^*(F_2)$. For each $v \in F_1^0$ and $e \in F_1^1$ set

$$Q_v = p_v$$

$$T_e = \begin{cases} s_e, & \text{if } e \neq e_0 \\ s_{e_{n+1}} s_{e_n} \cdots s_{e_1}, & \text{if } e = e_0. \end{cases}$$

We will show that $\{T_e, Q_v : e \in F_1^1, v \in F_1^0\}$ is Cuntz-Krieger F_1 -family that generates $C^*(F_2)$. It is clear that $Q_v Q_w = 0$ for all $v \neq w$. Let $e, f \in F_1^1$ with $e \neq f$. Then

$$T_e^* T_f = \begin{cases} s_e^* s_f, & \text{if } e \neq e_0 \text{ and } f \neq e_0 \\ s_{e_1}^* s_{e_2}^* \cdots s_{e_{n+1}}^* s_e, & \text{if } e = e_0 \\ s_e^* s_{e_{n+1}} s_{e_n} \cdots s_{e_1}, & \text{if } f = e_0 \end{cases}$$

$$= 0.$$

The last two cases is true because $e \neq e_{n+1}$ for all $e \in F_1^1$.

Now let $e \in F_1^1$. Then

$$T_e^* T_e = \begin{cases} s_e^* s_e, & \text{if } e \neq e_0 \\ s_{e_1}^* s_{e_2}^* \cdots s_{e_{n+1}}^* s_{e_{n+1}} s_{e_n} \cdots s_{e_1}, & \text{if } e = e_0 \end{cases}$$

$$= \begin{cases} p_{r_{F_2}(e)}, & \text{if } e \neq e_0 \\ p_{r_{F_2}(e_1)}, & \text{if } e = e_0 \end{cases}$$

$$= p_{r_{F_1}}(e)$$

$$= Q_{r_{F_1}(e)}$$

and

$$T_e T_e^* = \begin{cases} s_e s_e^*, & \text{if } e \neq e_0 \\ s_{e_{n+1}} s_{e_n} \cdots s_{e_1} s_{e_1}^* s_{e_2}^* \cdots s_{e_{n+1}}^*, & \text{if } e = e_0 \end{cases}$$

$$\leq \begin{cases} p_{s_{F_2}(e)}, & \text{if } e \neq e_0 \\ p_{s_{F_2}(e_{n+1})}, & \text{if } e = e_0 \end{cases}$$

$$= \begin{cases} p_{r_{F_1}(e)}, & \text{if } e \neq e_0 \\ p_{s_E(e_0)}, & \text{if } e = e_0 \end{cases}$$

$$= Q_{s_{F_1}(e)}.$$

Let $v \in F_1^0$ be a regular vertex. Note that $v \in F_2^0$ is a regular vertex. Suppose $v = v_i$ for some $i = 1, \dots, n$. Then $s_{F_2}^{-1}(v_i) = \{e_i\} = s_{F_1}^{-1}(v_i)$.

$$Q_v = Q_{v_i} = p_{v_i} = s_{e_i} s_{e_i}^* = T_{e_i} T_{e_i}^*.$$

Suppose $v \neq v_i$ for $i = 1, \dots, n$. We break this into two cases. Suppose $e_{n+1} \notin s_{F_2}^{-1}(v)$. Then $v \neq s_E(e_0)$. Since $v \neq v_i$ for $i = 1, \dots, n$ and $v \neq s_E(e_0)$, we have that $s_{F_1}^{-1}(v) \cap \{e_0, e_1, \dots, e_n\} = \emptyset$ and $s_{F_2}^{-1}(v) \cap \{e_1, \dots, e_n, e_{n+1}\} = \emptyset$. Thus,

$$s_{F_2}^{-1}(v) = s_E^{-1}(v) = s_{F_1}^{-1}(v).$$

Hence,

$$Q_v = p_v = \sum_{e \in s_{F_2}^{-1}(v)} s_e s_e^* = \sum_{e \in s_{F_1}^{-1}(v)} s_e s_e^* = \sum_{e \in s_{F_1}^{-1}(v)} T_e T_e^*.$$

Suppose $e_{n+1} \in s_{F_2}^{-1}(v)$. Then $v = s_{F_2}(e_{n+1}) = s_E(e_0)$, which implies that $e_0 \in s_{F_1}^{-1}(v)$. Note that $s_{e_i} s_{e_i}^* = p_{v_i}$ for all $i = 1, 2, \dots, n$. Thus,

$$\begin{aligned} Q_v = p_v &= \sum_{e \in s_{F_2}^{-1}(v)} s_e s_e^* \\ &= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} s_{e_{n+1}}^* \\ &= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} p_{v_n} s_{e_{n+1}}^* \\ &= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} s_{e_n} s_{e_n}^* s_{e_{n+1}}^* \\ &\vdots \\ &= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} s_{e_n} \cdots s_{e_1} s_{e_1}^* \cdots s_{e_n}^* s_{e_{n+1}}^* \\ &= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} T_e T_e^* + T_{e_0} T_{e_0}^* \\ &= \sum_{e \in s_{F_1}^{-1}(v)} T_e T_e^*. \end{aligned}$$

We have just shown that $\{T_e, Q_v : e \in F_1^1, v \in F_1^0\}$ is a Cuntz-Krieger F_1 -family. Hence, if $\{t_e, q_v : e \in F_1^1, v \in F_1^0\}$ is a universal Cuntz-Krieger F_1 -family generating $C^*(F_1)$, then there exists a $*$ -homomorphism $\psi : C^*(F_1) \rightarrow C^*(F_2)$ such that

$$\begin{aligned} \psi(q_v) &= Q_v \\ \psi(t_e) &= T_e \end{aligned}$$

for all $e \in F_1^1$ and $v \in F_1^0$.

Note that the only generator of $C^*(F_2)$ that is not included in

$$\{T_e, Q_v : e \in F_1^1, v \in F_1^0\}$$

is $s_{e_{n+1}}$. In this case, recall again that

$$p_{v_i} = s_{e_i} s_{e_i}^*$$

for all $i = 1, 2, \dots, n$. Therefore,

$$\begin{aligned} T_{e_0} T_{e_1}^* \dots T_{e_n}^* &= s_{e_{n+1}} s_{e_n} \dots s_{e_1} s_{e_1}^* \dots s_{e_n}^* \\ &= s_{e_{n+1}} s_{e_n} \dots s_{e_2} p_{v_1} s_{e_2}^* \dots s_{e_n}^* \\ &= s_{e_{n+1}} s_{e_n} \dots s_{e_2} p_{r_{F_2}(e_2)} s_{e_2}^* \dots s_{e_n}^* \\ &= s_{e_{n+1}} s_{e_n} \dots s_{e_2} s_{e_2}^* \dots s_{e_n}^* \\ &\vdots \\ &= s_{e_{n+1}} s_{e_n} s_{e_n}^* \\ &= s_{e_{n+1}} p_{v_n} \\ &= s_{e_{n+1}} p_{r_{F_2}(e_{n+1})} \\ &= s_{e_{n+1}}. \end{aligned}$$

Hence, $s_{e_{n+1}} \in \psi(C^*(F_1))$ which implies that ψ is surjective.

Note that the cycle structure of F_1 is determine by the cycle structure of E and vice versa. Moreover, the cycles of F_1 with no exits are in one-to-one correspondence to the cycles of F_2 with no exits. Let $\alpha = f_1 f_2 \dots f_m$ be a vertex-simple cycle in F_1 with no exits, (*vertex-simple* means that $s_{F_1}(f_i) \neq s_{F_1}(f_j)$ for $i \neq j$). Suppose $s_{F_1}(f_i) \neq s_{F_1}(e_0)$. Then α is a vertex-simple cycle in F_2 with no exits. Thus, s_α is a unitary in $C^*(F_2)$ with spectrum \mathbb{T} . Hence,

$$\psi(t_\alpha) = s_\alpha,$$

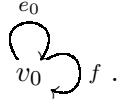
which implies that $\psi(t_\alpha)$ is a unitary in $C^*(F_2)$ with spectrum \mathbb{T} . Suppose $s_{F_1}(f_i) = s_{F_1}(e_0)$. Then $\alpha = e_0 f_2 \dots f_n$ since α is a vertex-simple cycle in F_1 with no exits. Note that

$$\psi(t_\alpha) = s_{e_{n+1}} s_{e_n} \dots s_{e_1} s_{f_2} \dots s_{f_n} = s_\beta$$

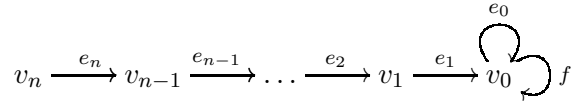
and $\beta = e_{n+1} e_n \dots e_1 f_2 \dots f_n$ is a vertex-simple cycle in F_2 with no exits. Hence, $\psi(t_\alpha) = s_\beta$ is a unitary in $C^*(F_2)$ with spectrum \mathbb{T} .

From the above paragraph and the fact that $\psi(q_v) = p_v \neq 0$ for all $v \in F_1^0$, by Theorem 1.2 of [15], ψ is injective. Therefore, ψ is an isomorphism. \square

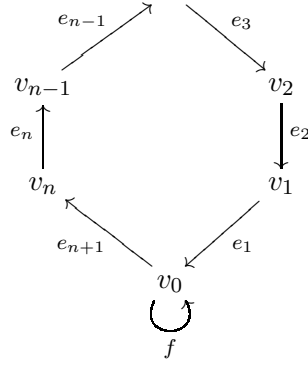
Remark 3.2. Let E be the graph



Then F_1 in the above proposition is the graph



and F_2 in the above proposition is the graph



Remark 3.3. The above proposition allows one to remove heads of finite length while preserving isomorphism classes.

Definition 3.4. Let E be a graph and let H be a hereditary subset of E^0 . Consider the set

$$F(H) = \{\alpha \in E^* : \alpha = e_1 e_2 \dots e_n, s_E(e_n) \notin H, r_E(e_n) \in H\}.$$

Let $\overline{F}(H)$ be another copy of $F(H)$ and we write $\overline{\alpha}$ for the copy of α in $\overline{F}(H)$. Define a graph $E(H)$ as follows:

$$E(H)^0 = H \cup F(H)$$

$$E(H)^1 = s_E^{-1}(H) \cup \overline{F}(H)$$

and we extend s_E and r_E to $E(H)$ by defining $s_{E(H)}(\overline{\alpha}) = \alpha$ and $r_{E(H)}(\overline{\alpha}) = r_E(\alpha)$.

Theorem 3.5. *Let E be a graph and let H be a hereditary subset of E^0 . Suppose*

$$(E^0 \setminus H, r_E^{-1}(E^0 \setminus H), r_E, s_E)$$

is a finite acyclic graph and for each $v \in E^0 \setminus H$, there exists a path in E from v to H . Assume furthermore that the set $s^{-1}(E^0 \setminus H) \cap r^{-1}(H)$ is finite. Then $C^(E) \cong C^*(E(H))$.*

Proof. Let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be a universal Cuntz-Krieger E -family generating $C^*(E)$. For $v \in E(H)^0$ define

$$Q_v := \begin{cases} p_v & \text{if } v \in H \\ s_{\alpha} s_{\alpha}^* & \text{if } v = \alpha \in F(H) \end{cases}$$

and for $e \in E(H)^1$ define

$$T_e := \begin{cases} s_e & \text{if } e \in s_E^{-1}(H) \\ s_{\alpha} & \text{if } e = \overline{\alpha} \in \overline{F}(H) \end{cases}$$

We shall show that $\{T_e, Q_v : e \in E(H)^1, v \in E(H)^0\}$ is a Cuntz-Krieger $E(H)$ -family in $C^*(E)$. To begin, we see that the Q_v 's are mutually orthogonal projections and the T_e are partial isometries with mutually orthogonal ranges. (The orthogonality follows from the fact that an element in $F(H)$ cannot extend an element in $F(H)$ and the fact that $s_E(\alpha) \notin H$ for all $\alpha \in F(H)$.)

To see the Cuntz-Krieger relations hold, we consider cases for $e \in E(H)^1$. If $e \in s_E^{-1}(H)$, then $r_H(e) \in H$ and

$$T_e^* T_e = s_e^* s_e = p_{r_E(e)} = Q_{r_{E(H)}(e)}.$$

If $e = \bar{\alpha} \in F(H)$, then $r_E(\alpha) \in H$ and

$$T_e^* T_e = T_{\bar{\alpha}}^* T_{\bar{\alpha}} = s_{\alpha}^* s_{\alpha} = p_{r_E(\alpha)} = Q_{r_E(\alpha)} = Q_{r_{E(H)}(\bar{\alpha})} = Q_{r_{E(H)}(e)}.$$

For the second Cuntz-Krieger relation, we again let $e \in E(H)^1$ and consider cases. If $e \in s_E^{-1}(H)$, then

$$Q_{s_{E(H)}(e)} T_e = p_{s_E(e)} s_e = s_e = T_e.$$

If $e = \bar{\alpha} \in \bar{F}(H)$, then

$$Q_{s_{E(H)}(e)} T_e = Q_{\alpha} T_{\bar{\alpha}} = s_{\alpha} s_{\alpha}^* s_{\alpha} = s_{\alpha} = T_{\bar{\alpha}} = T_e.$$

Thus $Q_{s_{E(H)}(e)} T_e = T_e$ for all $e \in E(H)^1$, so that $T_e T_e^* \leq Q_{s_{E(H)}(e)}$ for all $e \in E(H)^1$, and the second Cuntz-Krieger relation holds.

For the third Cuntz-Krieger relation, suppose that $v \in E(H)^0$ and that v is regular. If $v \in H$, then the set of edges that v emits in $E(H)$ is equal to the set of edges that v emits in E , and hence

$$Q_v = p_v = \sum_{\{e \in E^1 : s_E(e) = v\}} s_e s_e^* = \sum_{\{e \in E(H)^1 : s_{E(H)}(e) = v\}} T_e T_e^*.$$

If $v \in F(H)$, then $v = \alpha$ with $r_{E(H)}(\alpha) \in H$, and the element $\bar{\alpha}$ is the unique edge in $E(H)^0$ with source v , so that

$$Q_v = s_{\alpha} s_{\alpha}^* = T_{\bar{\alpha}} T_{\bar{\alpha}}^*.$$

Thus the third Cuntz-Krieger relation holds, and

$$\{T_e, Q_v : e \in E(H)^1, v \in E(H)^0\}$$

is a Cuntz-Krieger $E(H)$ -family in $C^*(E)$.

If $\{q_v, t_e : v \in E(H)^0, E(H)^1\}$ is a universal Cuntz-Krieger $E(H)$ -family generating $C^*(E(H))$, then by the universal property of $C^*(E(H))$ there exists a $*$ -homomorphism $\phi : C^*(E(H)) \rightarrow C^*(E)$ with $\phi(q_v) = Q_v$ for all $v \in E(H)^0$ and $\phi(t_e) = T_e$ for all $e \in E(H)^1$.

We shall show injectivity of ϕ , by applying the generalized Cuntz-Krieger uniqueness theorem of [15]. To verify the hypotheses, we first see that if $v \in E(H)^0$, then $\phi(q_v) = Q_v \neq 0$. Second, if $e_1 \dots e_n$ is a vertex-simple cycle in $E(H)$ (*vertex-simple* means $s(e_i) \neq s(e_j)$ for $i \neq j$) with no exits,

then since the cycles in $E(H)$ come from cycles in E all lying in the subgraph given by

$$(H, s_E^{-1}(H), s_E, r_E),$$

we must have that $e_i \in E^1$ for all $1 \leq i \leq n$, and $e_1 \dots e_n$ is a cycle in E with no exits. Thus $\phi(t_{e_1 \dots e_n}) = \phi(t_{e_1}) \dots \phi(t_{e_n}) = s_{e_1} \dots s_{e_n} = s_{e_1 \dots e_n}$ is a unitary whose spectrum is the entire circle. It follows from the generalized Cuntz-Krieger uniqueness theorem Theorem 1.2 of [15] that ϕ is injective.

We now show that ϕ is surjective. Let $e \in E^1$ such that $r_E(e) \in H$. If $s_E(e) \in H$, then

$$s_e = T_e = \phi(t_e) \in \text{im}(\phi).$$

Suppose $s_E(e) \notin H$. So, $e \in F(H)$. Hence,

$$s_e = T_{\bar{e}} = \phi(t_{\bar{e}}) \in \text{im}(\phi).$$

We now show that $s_e \in \text{im}(\phi)$ for all $e \in r_E^{-1}(E^0 \setminus H)$. By assumption, there is for every vertex $v \in E^0 \setminus H$ a path from v to H . Define for each k the subset D_k of $E^0 \setminus H$ as the set of vertices v for which k is the maximal length of a path from v to H . Put $D_0 = H$, and note that for $k \geq 1$, all vertices in D_k are regular. By induction over $k \geq 1$ we will show for every path α in E with $r_E(\alpha) \in D_k$ that $s_\alpha \in \text{im}(\phi)$.

For $k = 1$ and α a path in E with $r_E(\alpha) \in D_1$, we note that $r_E(e) \in H$ for all $e \in s_E^{-1}(r_E(\alpha))$, hence

$$\begin{aligned} s_\alpha &= s_\alpha p_{r_E(\alpha)} = \sum_{e \in s_E^{-1}(r_E(\alpha))} s_\alpha s_e s_e^* \\ &= \sum_{e \in s_E^{-1}(r_E(\alpha))} T_{\alpha \bar{e}} T_{\bar{e}}^* \\ &= \sum_{e \in s_E^{-1}(r_E(\alpha))} \phi(t_{\alpha \bar{e}} t_{\bar{e}}^*) \in \text{im}(\phi) \end{aligned}$$

since $\alpha \bar{e}, e \in F(H)$.

For $k > 1$ and α a path in E with $r_E(\alpha) \in D_k$, we note that for all $e \in s_E^{-1}(r_E(\alpha))$ there is a $j < k$ for which $r_E(\alpha e) = r_E(e) \in D_j$, hence

$$s_\alpha = \sum_{e \in s_E^{-1}(r_E(\alpha))} s_{\alpha e} s_e^* \in \text{im}(\phi).$$

We have just shown that $s_e \in \text{im}(\phi)$ for all $e \in E^1$. We now show that $p_v \in \text{im}(\phi)$ for all $v \in E^0$. Note that if $v \in E^0$ and v is not regular, then $v \in H$. Hence, $p_v = Q_v = \phi(q_v)$. Let $v \in E^0$ that is regular. Then for each $e \in s_E^{-1}(v)$, $s_e, s_e^* \in \text{im}(\phi)$. Therefore,

$$p_v = \sum_{e \in s_E^{-1}(v)} s_e s_e^* \in \text{im}(\phi).$$

Since $\{p_v, s_e : v \in E^0, e \in E^1\} \subseteq \text{im}(\phi)$ and $\{p_v, s_e : v \in E^0, e \in E^1\}$ generates $C^*(E)$ we have that ϕ is surjective. Therefore, ϕ is an isomorphism. \square

Corollary 3.6. *Let E be a graph, let $v_0 \in E^0$, and let $n \in \mathbb{N}$. Define a graph F_1 as follows:*

$$F_1^0 = E^0 \cup \{v_1, v_2, \dots, v_n\} \text{ and } F_1^1 = E^1 \cup \{e_1, e_2, \dots, e_n\}$$

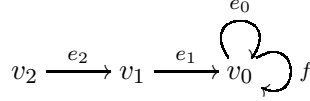
where r_{F_1} and s_{F_1} extends r_E and s_E respectively, and $r_{F_1}(e_i) = v_{i-1}$ and $s_{F_1}(e_i) = v_i$. Define a graph F_2 as follows:

$$F_2^0 = E^0 \cup \{v_1, v_2, \dots, v_n\} \text{ and } F_2^1 = E^1 \cup \{e_1, e_2, \dots, e_n\}$$

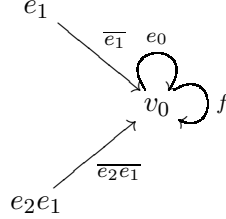
where r_{F_2} and s_{F_2} extends r_E and s_E respectively, and $r_{F_2}(e_i) = v_0$ and $s_{F_2}(e_i) = v_i$. Then $C^*(F_1) \cong C^*(F_2)$.

Proof. Note that E^0 is a hereditary subset of F_1^0 . Moreover $(F_1^0 \setminus E^0, r_{F_1}^{-1}(F_1^0 \setminus E^0), r_{F_1}, s_{F_1})$ is a finite acyclic graph and for each $v \in F_1^0 \setminus E^0$, there exists a path in F_1 from v to E^0 . Finally, $s_F^{-1}(F_1^0 \setminus E^0) \cap r_{F_1}^{-1}(E^0)$ is finite. Thus, Theorem 3.5, $C^*(F_1) \cong C^*(F_1(E^0))$. It is easy to see that there exists a graph isomorphism from $F_1(E^0)$ to F_2 . Thus, $C^*(F_1(E^0)) \cong C^*(F_2)$. \square

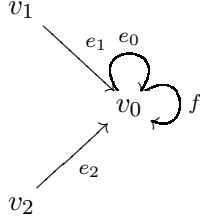
In above corollary, if F_1 is the graph



then $F_1(\{v_0\})$ is the graph



which is isomorphic to the graph F_2



Theorem 3.7. *Let E be a graph. Then the following are equivalent:*

- (1) E is finite graph with no sinks.
- (2) $C^*(E)$ is isomorphic to a Cuntz-Krieger algebra.

(3) $C^*(E)$ is unital and

$$\text{rank}(K_0(C^*(E))) = \text{rank}(K_1(C^*(E))).$$

Proof. Suppose E is a finite graph with no sinks. Remove the sources from E , and remove the vertices that then become sources; repeat this procedure finitely many times to get a sub-graph F of E that has no sinks and no sources. Notice that F^0 is a hereditary subset of E^0 , that

$$(E^0 \setminus F^0, r_E^{-1}(E^0 \setminus F^0), r_E, s_E)$$

is a finite acyclic graph, and that for each $v \in E^0 \setminus F^0$, there exists a path in E from v to F^0 . Therefore, by Theorem 3.5, $C^*(E) \cong C^*(E(F^0))$. We can apply Corollary 3.6 and Proposition 3.1 as many times as needed (but finitely many times), to get a finite graph E_1 with no sinks and no sources such that $C^*(E(F^0)) \cong C^*(E_1)$. Note that $C^*(E_1)$ is a Cuntz-Krieger algebra and $C^*(E) \cong C^*(E_1)$. We have just shown (1) implies (2).

Suppose $C^*(E)$ is isomorphic to a Cuntz-Krieger algebra. Then $C^*(E)$ is unital. Moreover, by the K -theory computation (Theorem 3.1 of [8]),

$$\text{rank}(K_0(C^*(E))) = \text{rank}(K_1(C^*(E))).$$

We have just shown (2) implies (3).

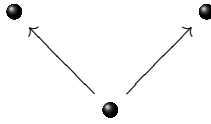
Suppose $C^*(E)$ is unital. Then E^0 is a finite set. Since

$$\text{rank}(K_0(C^*(E))) = \text{rank}(K_1(C^*(E))),$$

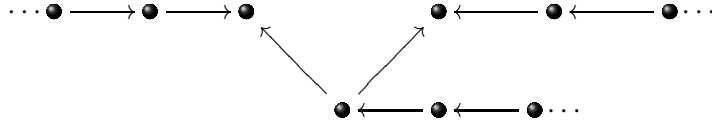
by the K -theory computation (Theorem 3.1 of [8]), E has no singular vertices. Hence, E is a finite graph with no sinks. Therefore, (3) implies (1). \square

Definition 3.8. Let E be a graph and let SE be the graph obtained by adding an infinite head to every vertex of E .

$E :$



$SE :$



We call SE the *stabilization* of E .

Theorem 3.9. Let E be a graph with finitely many vertices and let T be a finite hereditary subset of $(SE)^0$ such that $E^0 \subseteq T$. Set

$$p_T = \sum_{v \in T} p_v$$

where $\{s_e, p_v : e \in (SE)^1, (SE)^0\}$ is a universal Cuntz-Krieger SE -family generating $C^*(SE)$. Then p_T is a full projection in $C^*(SE)$ and there exists a sub-graph F of SE such that $C^*(F) \cong p_T C^*(SE) p_T$.

If in addition $C^*(E)$ is a Cuntz-Krieger algebra, then $p_T C^*(SE) p_T$ is a Cuntz-Krieger algebra.

Proof. The smallest saturated subset of $(SE)^0$ containing T is $(SE)^0$. Hence, p_T is a full projection.

Let $F = (T, s_{SE}^{-1}(T), r_{SE}, s_{SE})$. We claim that F is a complete subgraph of SE . It is clear that F is a subgraph of SE . We will show that $s_E^{-1}(v) = s_F^{-1}(v)$ for all $v \in F^0$. Let $v \in F^0$. Suppose $v \in E^0$. Then

$$s_{SE}^{-1}(v) = s_E^{-1}(v) = s_F^{-1}(v).$$

Suppose $v \in T \setminus E^0$. Then $s_{SE}^{-1}(v) = \{e\} = s_F^{-1}(v)$ for some e . Since F is a complete sub-graph of SE , we have that $C^*(F)$ isomorphic to the sub-algebra of $C^*(SE)$ generated by

$$\{p_v, s_e : s_{SE}(e), v \in T\},$$

which we denote by B . We claim that $p_T C^*(SE) p_T = B$.

Note that B is unital with unit p_T . Note that if $e \in s_{SE}^{-1}(T)$, $s_{SE}(e)$ and $r_{SE}(e)$ are elements of T . Therefore, for all $v \in T$ and all $e \in s_{SE}^{-1}(T)$,

$$p_v = p_T p_v p_T \in p_T C^*(SE) p_T$$

and

$$s_e = p_{s_{SE}(e)} s_e p_{r_{SE}(e)} = p_T p_{s_{SE}(e)} s_e p_{r_{SE}(e)} p_T \in p_T C^*(SE) p_T.$$

Hence, B is a sub-algebra of $p_T C^*(SE) p_T$.

Let α be a finite path in SE . Suppose $s_{SE}(\alpha)$ is not an element of T . Then

$$p_T p_{s_{SE}(\alpha)} = 0.$$

If $s_{SE}(\alpha) \in T$, then

$$p_T p_{s_{SE}(\alpha)} = p_{s_{SE}(\alpha)}.$$

From these observations, we get that

$$p_T s_\alpha s_\beta^* p_T = \begin{cases} s_\alpha s_\beta^*, & \text{if } s_{SE}(\alpha), s_{SE}(\beta) \in T \\ 0, & \text{otherwise.} \end{cases}$$

Since $e \in s_{SE}^{-1}(T)$ implies that $s_{SE}(e)$ and $r_{SE}(e)$ are elements of T , we have that α is a path in F if $s_{SE}(\alpha) \in T$. Therefore, if $s_{SE}(\alpha), s_{SE}(\beta) \in T$, then $s_\alpha, s_\beta^* \in B$. Hence,

$$p_T s_\alpha s_\beta^* p_T$$

is an element of B for all paths α and β in SE . We have just shown that $B = p_T C^*(SE) p_T$ which implies that $C^*(F) \cong B = p_T C^*(SE) p_T$.

Assume that $C^*(E)$ is isomorphic to a Cuntz-Krieger algebra. Then by Theorem 3.7, the graph E is finite and has no sinks. So since F is a graph obtained from the graph E by adding a finite head to some vertices of E^0 , the graph F is finite and with no sinks. By Theorem 3.7, $C^*(F)$ is a Cuntz-Krieger algebra. \square

4. UNITAL C^* -ALGEBRAS THAT ARE STABLY ISOMORPHIC TO CUNTZ-KRIEGER ALGEBRAS

Notation 4.1. In the following, for a C^* -algebra A and projections p in $M_n(A)$ and q in $M_m(A)$, $p \sim q$ denotes Murray-von Neumann equivalence between p and q in $M_\infty(A)$, i.e., the existence of an element v in $M_{m,n}(A)$ for which $p = v^*v$ and $q = vv^*$.

For a projection p in a C^* -algebra A and $n \in \mathbb{N}$, np will denote the projection

$$\underbrace{p \oplus p \cdots \oplus p}_{n\text{-times}} \in M_n(A).$$

Lemma 4.2. *Let E be a graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E -family generating $C^*(E)$. Let $v \in E^0$ and assume that v is regular. Then*

$$p_v \sim \sum_{e \in s^{-1}(v)} p_{r(e)}.$$

Proof. Follows directly from the Cuntz-Krieger relations, cf. Definition 2.1. \square

Lemma 4.3. *Let E be a row-finite graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E -family generating $C^*(E)$. Let $v, w \in E^0$ with $v \neq w$. If there is a path from v to w in E , then there exists a family $(m_u(v, w))_{u \in E^0}$ of non-negative integers satisfying*

$$p_v \sim p_w + \sum_{u \in E^0} m_u(v, w) p_u$$

with all but finitely many $m_u(v, w)$ equal 0. Moreover, $m_v(v, w)$ can be chosen such that

$$m_v(v, w) \geq |\{e \in E^1 \mid s_E(e) = r_E(e) = v\}|.$$

Proof. Let $e_1 \cdots e_n$ denote a path in E from v to w , i.e., $e_1, \dots, e_n \in E^1$ satisfying $r(e_i) = s(e_{i+1})$ for all $i \in \{1, \dots, n-1\}$, $s(e_1) = v$, and $r(e_n) = w$. Define $v_i = r(e_i)$ for $i \in \{1, \dots, n\}$, and $v_0 = v$. Then by Lemma 4.2,

$$\begin{aligned} p_v &\sim p_{v_1} + \sum_{e \in s^{-1}(v) \setminus \{e_1\}} p_{r(e)} \\ &\sim p_{v_2} + \sum_{e \in s^{-1}(v_1) \setminus \{e_2\}} p_{r(e)} + \sum_{e \in s^{-1}(v_0) \setminus \{e_1\}} p_{r(e)} \end{aligned}$$

$$\begin{aligned} & \vdots \\ & \sim p_w + \sum_{i=1}^{n-1} \left(\sum_{e \in s^{-1}(v_{i-1}) \setminus \{e_i\}} p_{r(e)} \right). \end{aligned}$$

Define $(m_u(v, w))_{u \in E^0}$ as the non-negative integer scalars in the above linear combination of $(p_u)_{u \in E^0}$, i.e., such that

$$\sum_{i=1}^{n-1} \left(\sum_{e \in s^{-1}(v_{i-1}) \setminus \{e_i\}} p_{r(e)} \right) = \sum_{u \in E^0} m_u(v, w) p_u.$$

This defines $(m_u(v, w))_{u \in E^0}$ for any pair $v, w \in E^0$ for which there is a path from v to w .

The last statement is clear from the construction of $m_u(v, w)$. \square

Theorem 4.4 (Theorem 3.5 of [2]). *Let E be a row-finite graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E -family generating $C^*(E)$. Then all projections in $C^*(E) \otimes \mathbb{K}$ are up to Murray-von Neumann equivalence of the form $\sum_{u \in E^0} m_u p_u$ with all but finitely many m_u equal 0.*

Proof. This is just a consequence of Theorem 3.5 of [2]. \square

Lemma 4.5. *Let E be a graph such that for each $v \in E^0$, there exists $e \in E^1$ such that $s_E(e) = r_E(e) = v$, i.e., every vertex of E is the based point of at least one cycle of length one. Then every hereditary subset H of E^0 is saturated.*

Proof. Suppose v is a regular vertex of E such that $r_E(s_E^{-1}(v)) \subseteq H$. By assumption there exists $e \in s_E^{-1}(v)$ such that $v = r_E(e) = s_E(e)$. Hence, $v \in r_E(s_E^{-1}(v))$ which implies that $v \in H$. \square

Lemma 4.6. *Let E be a finite graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E -family generating $C^*(E)$. Assume that E has no sinks and no sources, and every vertex of E is a based point of at least one cycle of length one.*

Let p be a norm-full projection in $C^(E) \otimes \mathbb{K}$. Then there exists a family $(m_u)_{u \in E^0}$ of integers satisfying*

$$p \sim \sum_{u \in E^0} m_u p_u$$

and $m_u \geq 1$ for all $u \in E^0$

Proof. By Theorem 4.4, there exists a family $(n_u)_{u \in E^0}$ of non-negative integers satisfying

$$p \sim \sum_{u \in E^0} n_u p_u.$$

Set $S_0 = \{u \in E^0 \mid n_u \neq 0\}$ and let H_0 be the smallest hereditary subset of E^0 that contains S_0 . By Lemma 4.5, H_0 is saturated. Note that $p \in I_{H_0}$ since $p \sim \sum_{u \in E^0} m_u p_u$. Hence, $I_{H_0} = C^*(E)$ which implies that $H_0 = E^0$. Therefore, for every $w \in E^0$, there exists $v \in S_0$ such that $v \geq w$.

Set $E^0 \setminus S_0 = \{w_0, w_1, \dots, w_m\}$. Let $v \in S_0$ such that $v \geq w_0$. By Lemma 4.3,

$$p_v \sim p_{w_0} + \sum_{u \in E^0} m_u(v, w_0) p_u$$

where $m_u(v, w_0) \geq 0$ and

$$m_v(v, w_0) \geq |\{e \in E^1 \mid s_E(e) = r_E(e) = v\}| \geq 1.$$

Therefore,

$$p \sim \sum_{u \in E^0} n'_u p_u$$

where $n'_u \geq 0$ for all $u \in E^0$. Moreover,

$$S_0 \subsetneq \{u \in E^0 \mid n'_u \neq 0\} = S_1$$

since $n'_{w_0} \neq 0$ but $n_{w_0} = 0$. Therefore, $|E^0 \setminus S_1| < |E^0 \setminus S_0|$.

Let H_1 be the smallest hereditary subset of E^0 that contains S_1 . Note that $E^0 = H_0 \subseteq H_1 \subseteq E^0$. Hence, $H_1 = E^0$. Hence, for each $w \in E^0$, there exists a $v \in S_1$ such that $v \geq w$. Therefore, we may continue this process to get a family $(m_u)_{u \in E^0}$ of non-negative integers satisfying

$$p \sim \sum_{u \in E^0} m_u p_u$$

and $m_u \geq 1$ for all $u \in E^0$. □

Proposition 4.7. *Let E be a finite graph with no sinks and no sources, and assume that every vertex of E is a based point of at least one cycle of length one. Let p be a norm-full projection in $C^*(E) \otimes \mathbb{K}$. Then there exists a finite graph F that has no sinks and no sources such that $C^*(F) \cong p(C^*(E) \otimes \mathbb{K})p$.*

Proof. Let SE be the graph obtained by adding infinite heads to every $v \in E^0$. Let $\{e_{ij}\}_{i,j}$ be a system of matrix units for \mathbb{K} . By Proposition 9.8 of [1] and its proof, there exists an isomorphism $\phi: C^*(E) \otimes \mathbb{K} \rightarrow C^*(SE)$ such that

$$K_0(\phi)([p_v \otimes e_{11}]) = [p_v]$$

for all $v \in E^0$. Let p be a norm-full projection in $C^*(E) \otimes \mathbb{K}$. By Lemma 4.6, p is Murray-von Neumann equivalent to $\sum_{u \in E^0} m_u p_u$ with $m_u \geq 1$ for all $u \in E^0$. Therefore, since $C^*(SE)$ has weak cancellation by Corollary 7.2 of [2], $\phi(p)$ is Murray-von Neumann equivalent to $p_T \in C^*(SE)$ such that T is a finite, hereditary subset of $(SE)^0$ with $E^0 \subseteq T$. By Theorems 3.9 and 3.7, $p_T C^*(SE) p_T \cong C^*(F)$ for some finite graph F with no sinks and

no sources. Note that $p(C^*(E) \otimes \mathbb{K})p \cong \phi(p)C^*(SE)\phi(p) \cong p_T C^*(SE)p_T$. Therefore, $p(C^*(E) \otimes \mathbb{K})p \cong C^*(F)$. \square

The following theorem answers a question asked by George A. Elliott at the NordForsk Closing Conference at the Faroe Islands, May 2012.

Theorem 4.8. *Let A be a unital C^* -algebra.*

- (1) *If A is stably isomorphic to a Cuntz-Krieger algebra, then A is isomorphic to a Cuntz-Krieger algebra.*
- (2) *Let A be a unital, nuclear, separable C^* -algebra with finitely many ideals and let $X = \text{Prim}(A)$. If $A \otimes \mathcal{O}_\infty$ is KK_X -equivalent to a Cuntz-Krieger algebra with real rank zero and primitive ideal space X , then $A \otimes \mathcal{O}_\infty$ is isomorphic to a Cuntz-Krieger algebra of real rank zero.*

Proof. We first prove (1). Let B be a Cuntz-Krieger algebra such that $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Note that $B = C^*(F)$ such that F is a finite graph with no sinks and no sources. By Theorem 5.2 of [14], collapsing a regular vertex that is not a based point of a cycle of length one preserves stable isomorphism classes. Therefore, since F is a finite graph with no sinks and no sources, we can apply Theorem 5.2 of [14] a finite number of times to get a finite graph E with no sinks and no sources, and every vertex of E is a based point of at least one cycle of length one, such that $C^*(F) \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. Hence, $A \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. Denote this isomorphism by ϕ .

Let $\{e_{ij}\}_{i,j}$ be a system of matrix units for \mathbb{K} . Since $1_A \otimes e_{11}$ is a norm-full projection in $A \otimes \mathbb{K}$, $p = \phi(1_A \otimes e_{11})$ is a norm-full projection in $C^*(E) \otimes \mathbb{K}$. By Proposition 4.7, $p(C^*(E) \otimes \mathbb{K})p$ is isomorphic to a Cuntz-Krieger algebra. Note $(1_A \otimes e_{11})(A \otimes \mathbb{K})(1_A \otimes e_{11}) \cong p(C^*(E) \otimes \mathbb{K})p$ and $A \cong (1_A \otimes e_{11})(A \otimes \mathbb{K})(1_A \otimes e_{11})$. Therefore, A is isomorphic to a Cuntz-Krieger algebra.

We will now use (1), to prove (2). Let B be a Cuntz-Krieger algebra with real rank zero such that $A \otimes \mathcal{O}_\infty$ is KK_X -equivalent to B and $\text{Prim}(B) \cong X$. By Folgerung 4.3 of [11], $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Therefore, $A \otimes \mathcal{O}_\infty$ is a unital C^* -algebra stably isomorphic to a Cuntz-Krieger algebra with real rank zero. By (1), we have that $A \otimes \mathcal{O}_\infty$ is isomorphic to a Cuntz-Krieger algebra. Since $A \otimes \mathcal{O}_\infty$ is stably isomorphic to a C^* -algebra with real rank zero, $A \otimes \mathcal{O}_\infty$ has real rank zero. Therefore, $A \otimes \mathcal{O}_\infty$ is isomorphic to a Cuntz-Krieger algebra with real rank zero. \square

Corollary 4.9. *Let A be a C^* -algebra. Then the following are equivalent.*

- (1) *A is a Cuntz-Krieger algebra.*
- (2) *$M_n(A)$ is a Cuntz-Krieger algebra for all $n \in \mathbb{N}$.*
- (3) *$M_n(A)$ is a Cuntz-Krieger algebra for some $n \in \mathbb{N}$.*

Proof. (1) implies (2) follows from Theorem 4.8. (2) implies (3) is obvious. Suppose $M_n(A)$ is a Cuntz-Krieger algebra for some $n \in \mathbb{N}$. In particular, $M_n(A)$ is a unital C^* -algebra with $1_{M_n(A)} = [x_{ij}]$. A computation shows that x_{11} is a multiplicative identity for A . Therefore, A is a unital C^* -algebra.

Since $A \otimes \mathbb{K} \cong M_n(A) \otimes \mathbb{K}$ and since $M_n(A)$ is a Cuntz-Krieger algebra, by Theorem 4.8, A is a Cuntz-Krieger algebra. \square

Corollary 4.10. *Let A be a Cuntz-Krieger algebra.*

- (1) *If p is a non-zero projection in A , then pAp is isomorphic to a Cuntz-Krieger algebra.*
- (2) *If p is a non-zero projection in $A \otimes \mathbb{K}$, then $p(A \otimes \mathbb{K})p$ is isomorphic to a Cuntz-Krieger algebra.*

Proof. We first prove (1) in the case when p is a norm-full projection. Suppose p is a norm-full projection. By Corollary 2.6 of [6], $pAp \otimes \mathbb{K} \cong A \otimes \mathbb{K}$. Therefore, pAp is a unital C^* -algebra that is stably isomorphic to a Cuntz-Krieger algebra. By Theorem 4.8, pAp is isomorphic to a Cuntz-Krieger algebra.

We now prove the general case in (1). Let $A = C^*(E)$ where E is a finite graph with no sinks and no sources. Let p be a non-zero projection of A . Set

$$I = \text{the ideal in } C^*(E) \text{ generated by } p.$$

Note that $pAp \subseteq I$ which implies that $pAp \subseteq pIp$. Since $pIp \subseteq pAp$, we have that $pAp = pIp$. Thus, pIp is a norm-full hereditary sub-algebra of I . By Corollary 2.6 of [6], $pIp \otimes \mathbb{K} \cong I \otimes \mathbb{K}$.

Since I is generated by a projection p , by Theorem 7.3 and the proof of Theorem 5.3 of [2], I is a gauge-invariant ideal of $C^*(E)$. Thus, by Theorem 3.7 of [3], there exists a hereditary saturated subset H of E^0 such that

$$I_H = \text{the ideal in } C^*(E) \text{ generated by } \{p_v \mid v \in H\}$$

is I .

Let $E_H = (H, s_E^{-1}(H), r_E, s_E)$. By Proposition 3.4 of [3], $I_H \otimes \mathbb{K} \cong C^*(E_H) \otimes \mathbb{K}$. Note that E_H is a finite graph with no sinks. By Proposition 3.1 of [14], we may continue to remove the sources to obtain a finite graph F with no sinks and no sources such that $C^*(E_H) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$. Hence, $C^*(F)$ is a Cuntz-Krieger algebra and $pAp = pIp$ is a unital C^* -algebra that is stably isomorphic to $C^*(F)$. By Theorem 4.8, pAp is isomorphic to a Cuntz-Krieger algebra.

We now prove (2). Let p be a non-zero projection in $A \otimes \mathbb{K}$. Recall that $A = C^*(E)$, where E is a finite graph with no sinks and no sources. By Theorem 4.4, there exists a non-empty subset S of E^0 and a collection of positive integers $\{m_v\}_{v \in S}$ such that p is Murray-von Neumann equivalent to $\sum_{v \in S} m_v p_v$. Set $q = \sum_{v \in S} p_v$. Then q is a non-zero projection in A and by (1), we have that $qAq \cong C^*(F)$ for some finite graph F with no sinks and no sources. By Theorem 5.3 of [2], p and $q \otimes e_{11}$ generate the same ideal of $A \otimes \mathbb{K}$. Hence, $qAq \otimes \mathbb{K} \cong (q \otimes e_{11})A \otimes \mathbb{K}(q \otimes e_{11}) \cong p(A \otimes \mathbb{K})p \otimes \mathbb{K}$. Therefore, $p(A \otimes \mathbb{K})p$ is stably isomorphic to a Cuntz-Krieger algebra. By Theorem 4.8, $p(A \otimes \mathbb{K})p$ is isomorphic to a Cuntz-Krieger algebra. \square

Corollary 4.11. *Let A be a Cuntz-Krieger algebra. If p is a projection in $A \otimes \mathbb{K}$, then $p(A \otimes \mathbb{K})p$ is semiprojective. If p is a projection in A , then pAp is semiprojective.*

Proof. This follows from Corollary 4.10 since by Corollary 2.24 of [5] all Cuntz-Krieger algebras are semiprojective and by Corollary 2.29 of [5] all stabilized Cuntz-Krieger algebras are semiprojective. \square

5. ACKNOWLEDGEMENTS

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